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Hydrodynamic Stability of Thin Liquid Films Flowing Down an Inclined Plane with Accompanying Heat Transfer and Interfacial Shear

SIU-MING YIH
and
RICHARD C. SEAGRAVE
Department of Chemical Engineering
Engineering Research Institute
Iowa State University
Ames, Iowa 50011

An analysis is made on the influence of heat transfer and interfacial shear on the hydrodynamic stability of thin liquid films flowing down an inclined plane. The linear stability of the flow is determined by a successive perturbation solution of the governing Orr-Sommerfeld equation modified to include the effects of temperature on viscosity. Stability criteria are presented which show that a cooled wall is destabilizing, while a heated wall is stabilizing. The surface perturbation stresses of the concurrent gas shear are also shown to be destabilizing factors. Neutral stability curves are presented and compared with the isothermal case.

SCOPE

Falling films are usually employed as a heat or mass exchange medium in industrial equipment such as vertical condensers, film evaporators, and absorption towers. Since surface waves appear at small Reynolds numbers for film flow and enhance the momentum, heat, and mass transfer rates considerably over those predicted for smooth laminar flow, it is often desirable, for process design and prediction purposes, to be able to assess a priori the critical conditions under which waves will appear and to study the stability chracteristics of these liquid films under various operating conditions. The hydrodynamic stability of an isothermal liquid film flowing down an inclined plane was first treated in detail by Benjamin (1957) and Yih (1963). Subsequent workers have followed these two lines with slight modifications, such as incorporating the influence of airflow (Craik, 1966; Smith, 1970), surface contamination (Whitaker, 1964; Smith, 1970), evaporative flux (Bankoff, 1971), and condensation (Marschall and Lee, 1973). Much less, however, is known about the effects of heat transfer and interfacial shear on surface wave instability. The objective of the present study is to establish stability criteria for liquid films that are flowing down an inclined plane and are influenced by heat transfer and interfacial shear. In particular, the effect of variation of viscosity with temperature, the importance of which has

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been demonstrated by studies on shear wave instability (Potter and Graber, 1972; Wazzan et al., 1968) is taken into account. Stability criteria are developed by a method similar to Yih's (1963). The similarities and differences in heat transfer influence on shear wave and surface wave instability are compared.

CONCLUSIONS AND SIGNIFICANCE

The linear stability of a liquid film flowing down an inclined plane with accompanying heat transfer and interfacial shear is determined by a method using small disturbance theory. The governing Orr-Sommerfelt (O-S) equation, which is modified to include the effects of temperature on viscosity, is solved by a successive perturbation method. The analysis shows that wall cooling is destabilizing because of the lower viscosity generated at the film surface, while the reverse is true for wall heating. The additional viscosity gradient terms in the modified O-S equation are shown to be important destabilizing factors. The stability of the flow is also influenced by the variation of the mean velocity profile in the liquid film. Thus, the effects of heat transfer on surface wave instability in falling films are, in many respects, similar to the shear wave instability in other types of flow. The neutral stability curves show that the angle of inclination is a stabilizing factor, and the critical Reynolds number for surface wave formation is more influenced by wall heating than wall cooling. Interfacial shear is also shown to be a destabilizing factor because of the generation of surface perturbation stresses. However, the destabilizing effects of interfacial shear and wall cooling are not simply additive because of the coupling and interaction in the mean velocity profile. The linear stability analysis enables one to determine the critical conditions under which waves will appear and has important application to processes dealing with heat and mass transfer in falling films.

It is a well-known fact that surface waves appear on a falling film throughout almost the entire range of practical flow situations except at low flow rates, so it is important for industrial process design and prediction purposes to predict the onset and growth of these waves under different flow conditions.

The earliest study by Kapitza (1948) shows that a critical Reynolds number R_{cr} exists which depends on the kinematic surface tension, the viscosity of the liquid, and the angle of inclination. Rigorous formulation and solution for the linear stability of a liquid film flowing down an inclined plane under the influence of gravity and surface tension were first treated in detail by Benjamin (1957) and Yih (1963). A number of approximate analytical and numerical solutions for the same problem have also appeared in the literature (Anshus and Goren, 1966; Whitaker, 1964). The majority of the experimental data on wave properties has been taken under room conditions, where the most highly amplified waves appear. Krantz and Goren (1971) measured wavelengths, wave celerities, and growth rates of two-dimensional waves under imposed disturbances of controlled amplitude and frequency for low Reynolds number film flow. Their results seem to confirm, at least semiquantitatively, that the linear stability and momentum integral theories are valid for small amplitude long waves on falling films flowing at small Reynolds number. Craik (1966) studied the stability of a horizontal liquid film under the influence of air flow. It was found that the tangential stress perturbation component was a dominant destabilizing factor in very thin films. Smith (1970) investigated the influence of wind stress and surface active agents on three-dimensional film flow down an inclined plane. Surface contamination was found to be a stabilizing factor in agreement with other previous investigations.

A limited number of studies have been performed on the influence of heat transfer on shear wave instability. The stability of laminar boundary-layer flow with heat transfer was studied by Wazzan et al. (1968). The equation that governs the stability of the flow to shear wave (Tollmein-Schlichting wave) disturbances is a modified Orr-Sommerfeld (O-S) equation including the effect of temperature variation on viscosity. Numerical calculations for water show that for cooling of the boundary layer, the critical Reynolds number is reduced; for heating, the critical Reynolds number increases until it reaches a maximum and then decreases with further heating. It was concluded that the velocity profile had a great influence on the flow stability and that the viscosity derivative terms in the modified Orr-Sommerfeld equation had a destabilizing effect. Potter and Graber (1972) extended the above heat transfer analyses to plane Poiseuille flow. They showed that a temperature difference between the two walls was always a destabilizing factor, with the critical Reynolds number reduced below that of isothermal conditions. They further showed that even when the viscosity gradient terms were small, their inclusion was extremely important for liquids.

Although some investigations have been presented on the influence of heat transfer on shear wave instability, relatively few have appeared in the literature concerning the stability of falling films undergoing heat transfer. Besides film condensation and evaporation, significant temperature gradients may exist, for example, in some gas absorption processes such as the sulfonation and sulfation of an organic acid in a wetted wall column, where the reaction between the organic acid with sulfur trioxide is highly exothermic (Vander Mey, 1967). The color of the product, which is very sensitive to temperature, has to be controlled carefully by cooling through the walls. The mass transfer rate is affected substantially by heat transfer because the liquid viscosity, molecular diffusivity, kinetic rate constant, and gas solubility are all functions of temperature. In addition, heat transfer may induce surface wave instability and cause the mass transfer rate to be increased by two to threefold. Marschall and Lee (1973) have analyzed the two-dimensional instability of a condensate film flowing down a vertical wall by asssuming that the mean temperature profile is linear, and the mean velocity profile is semiparabolic. The Orr-Sommerfeld equation remains unchanged, while the normal stress boundary condition is modified to include a balance between the vapor velocity and the heat flux at the interface. It was shown that the condensation mass transfer had a stabilizing effect on the film for a constant temperature drop, while no stabilizing effect could be found for the case of constant heat flux. Bankoff (1971) considered the evaporative flux as influencing only the pressure boundary condition at the interface and carried out an analysis similar to Yih's (1963). He showed that evaporation had a destabilizing effect on the film. Both of the investigations above have neglected the variation of physical properties with temperature. Since, in evaporation or condensation processes, large temperature gradients exist in the film, the assumption of constant property flow can hardly be justified, especially when the liquid viscosity is a strong function of temperature.

GENERAL FORMULATION

The system considered is one of a two-dimensional plane parallel flow under the action of gravity. Heat transfer occurs as a linear temperature drop across the film, with T_0 at the wall and T_1 at the gas-liquid interface. All the physical properties are assumed to be a negligible function of temperature with the exception of the liquid viscosity which is assumed to follow the exponential temperature relationship. The gas flow exerts a constant tangential stress on the free surface, the latter being restrained also by surface tension. Employing the same assumptions as used by Shair (1971), the viscosity-temperature dependence is of the form, for $(T_1 - T_0)/T_0 << 1$

$$\mu = \mu_0 e^{-\alpha\eta} \tag{1}$$

Values of the parameter E_a have been determined experimentally by Iyer (1930), who examined eighty-seven different liquids of diverse chemical constitution. The exponential dependency was found to be quite adequate in describing the temperature variation on liquid viscosity for a variety of pure liquids. Increasing values of α therefore denote increasing variation of viscosity with temperature across the film. Typical values of α lie between 0 and 1. The dimensionless velocity profile of the primary flow is given by Shair as

$$\overline{U} = \frac{U}{<U>} = \left\{ (1-\beta)\left[\frac{\alpha\eta e^{\alpha\eta} - \alpha e^{\alpha\eta} - e^{\alpha\eta} + \alpha + 1}{1 + \alpha - e^{\alpha}}\right] + \beta\left[\frac{e^{\alpha\eta} - 1}{e^{\alpha} - 1}\right]\right\} \Big/ z_1 \tag{2}$$

where

$$z_1 = (1-\beta)\left[\frac{\alpha^2 + 2\alpha + 2 - 2e^{\alpha}}{\alpha^2 + \alpha - \alpha e^{\alpha}}\right] + \beta\left[\frac{e^{\alpha} - \alpha - 1}{\alpha e^{\alpha} - \alpha}\right] \tag{3}$$

The value of $\beta = 0$ corresponds to the freely falling film, while $\beta = 1$ corresponds to the plane Couette flow. If an arbitrarily small two-dimensional harmonic disturbance is imposed on the primary flow, the disturbance will either grow or decay in time or space. The condition in which there is no growth or decay defines the condition for neutral stability. The usual assumptions are that the Fourier components of an arbitrary disturbance are dynamically independent and that it is sufficient to consider only two-dimensional disturbances if any three-dimensional disturbance is represented by the same equations as a two-dimensional disturbance. In the presence of gas flow, surface perturbation stresses are formed by the interaction of the mean gas flow with the harmonic disturbance at the gas-liquid interface.

Let

$$u_1 = \overline{U} + u', \quad v_1 = v', \quad p_1 = \overline{P} + p', \quad \hat{T} = \overline{T} + T' \tag{4}$$

where the primed quantities represent perturbations and $\overline{U}$, $\overline{P}$, $\overline{T}$ represent the velocity, pressure, and temperature of the primary flow, respectively.

It is assumed that the stream function for an infinitesimal disturbance is given in dimensionless form as

$$\psi = \phi(\eta) \exp[ik(x - c\tau)] \tag{5}$$

$$p' = f(\eta) \exp[ik(x - c\tau)] \tag{6}$$

$$T' = \hat{F}(\eta) \exp[ik(x - c\tau)] \tag{7}$$

By defining

$$u' = \psi_\eta, \quad v' = \psi_x \tag{8}$$

the continuity equation is automatically satisfied. Equation (8) is substituted into the equations of motion. After linearization and elimination of $f'(\eta)$, a fourth-order, linear, homogeneous, ordinary differential equation is obtained which is the modified Orr-Sommerfeld equation with the inclusion of the effect of variable viscosity (henceforth referred to as the modified O-S equation):

$$ikR[(\overline{U} - c)(\phi'' - k^2\phi) - \overline{U}''\phi] = \tilde{\mu}(\phi'''' - 2k^2\phi'' + k^4\phi) + 2\frac{d\tilde{\mu}}{d\eta}(\phi''' - k^2\phi') + \frac{d^2\tilde{\mu}}{d\eta^2}(\phi'' + k^2\phi) \tag{9}$$

This equation has also been used to study shear wave instability for plane Poiseuille flow and boundary-layer flow under heat transfer. The modified O-S equation contains extra viscosity derivative terms arising from the effect of variable viscosity and is particularly important for the study of liquids.

Since the velocity perturbations vanish at the wall, the boundary conditions at the wall are

$$(1)\, u'(0) = \psi_\eta(0) = 0 \tag{10}$$

$$(2)\, v'(0) = -\psi_x(0) = 0 \tag{11}$$

The two boundary conditions at the gas-liquid interface concern the balance of the tangential and normal stresses. Following a similar procedure by Craik (1966) and Smith (1970) in defining stress perturbation components for airflow, the tangential and normal stress boundary conditions at the surface $\eta = 1 + a$ are, respectively

$$(3)\, \tilde{\mu}\left[\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial \eta}\right] = \frac{(\tau_1 + \tau_t a)\Delta}{\mu_0 <U>} \tag{12}$$

$$(4)\, -p_1 + \frac{2}{R}\tilde{\mu}\frac{\partial v_1}{\partial \eta} - S\frac{\partial^2 a}{\partial x^2} = \frac{\tau_n a}{\rho_1 <U>^2} \tag{13}$$

Equations (12) and (13) apply only at $\eta = 1 + a$ and have to be linearized so that they apply at $\eta = 1$. The four boundary conditions for the modified O-S equation can be expressed as

$$(1)\, \phi'(0) = 0 \tag{14}$$

$$(2)\, \phi(0) = 0 \tag{15}$$

$$(3)\, e^{-\alpha}\left[\phi''(1) + \left(k^2 + \frac{\overline{U}''|_{\eta=1}}{c - \overline{U}(1)}\right)\phi(1)\right] = R\Sigma\frac{\phi(1)}{c - \overline{U}(1)} \tag{16}$$

$$(4)\, -\left\{k\left[\frac{\frac{R\cos\theta}{F^2} + k^2RS}{c - \overline{U}(1)} - R\overline{U}'(1) - \frac{R\Pi}{c - \overline{U}(1)}\right]\right\}\phi(1) + k[R(c - \overline{U}(1)) + 3ike^{-\alpha}] \cdot \phi'(1) - ie^{-\alpha}\phi'''(1) + i\alpha e^{-\alpha}[\phi''(1) + k^2\phi(1)] = 0 \tag{17}$$

A remaining condition, the linearized kinematic condition at the gas-liquid interface, is

$$v' = -\psi_x = a_\tau + \overline{U}a_x \tag{18}$$

or

$$a = \left[\frac{\phi(1)}{c - \overline{U}(1)}\right]\exp[ik(x - c\tau)] \tag{19}$$

Equation (9), together with Equations (14) to (17), constitute a complete eigenvalue problem. For temporal growth, the complex wave velocity c is solved in terms of the wave number and other governing parameters. The harmonic disturbance of wave number k will grow and amplify in time if c_i is positive and damp, or decay when c_i is negative. The equation

$$c_i(R, F, k, \theta, S, \alpha, \beta, \Sigma, \Pi) = 0 \quad (20)$$

depicts a relation between the wave number and other parameters that influence the flow and is defined as the neutral stability condition. The energy equation and its associated boundary conditions can be similarly manipulated. Because the modified O-S equation and the boundary conditions do not depend on the energy equation, the stability criteria can be obtained without solving the energy equation.

Since the commonly observed wavelengths of the harmonic disturbances on liquid films are usually large compared to the film thickness, the wave number can be assumed to be small. Provided $k^2 << 1$ and $kR << 1$, a successive perturbation method can be used to solve the modified O-S equation. Yih (1963) has shown that only two perturbations will be accurate enough to solve the governing O-S equation at small wave numbers and to obtain the condition for neutral stability. His approach will be used here.

SOLUTION

In the first approximation, k is set equal to zero, and the modified O-S equation becomes

$$\tilde{\mu}\phi'''' + 2\frac{d\tilde{\mu}}{d\eta}\phi''' + \frac{d^2\tilde{\mu}}{d\eta^2}\phi'' = 0 \quad (21)$$

Since $\tilde{\mu} = e^{-\alpha\eta}$, this is equivalent to solving

$$\phi'''' - 2\alpha\phi''' + \alpha^2\phi'' = 0 \quad (22)$$

which is a fourth-order, linear, differential equation with constant coefficients. The boundary conditions are

$$(1)\phi'(0) = 0 \quad (23)$$

$$(2)\phi(0) = 0 \quad (24)$$

$$(3)\phi''(1) + \frac{\overline{U}''|_{\eta=1}}{c - \overline{U}(1)}\phi(1) = 0 \quad (25)$$

$$(4)\phi'''(1) - \alpha\phi''(1) = 0 \quad (26)$$

where Σ is neglected in the first approximation. Craik (1966) has shown Σ to be on the order of $k^{5/3}$ for flow over a rigid boundary, but it is included here in the first-order approximation. A general solution of Equation (22) can easily be shown to be

$$\phi = \overline{A} + \overline{B}\eta + \overline{C}e^{\alpha\eta} + \overline{D}\eta e^{\alpha\eta} \quad (27)$$

Boundary condition Equation (26) demands $\overline{D} = 0$. Applying the remaining boundary conditions with suitable substitution results in $\overline{B} = \alpha$, $\overline{C} = -1$, and

$$c_0 = \frac{(2 - \beta) + \beta\dfrac{(1 + \alpha - e^\alpha)}{e^\alpha - 1}}{Z_1} \quad (28)$$

Since the eigenfunction is determined only up to a multiplicative constant, $\overline{A}$ can be arbitrarily set equal to one, and the first approximation for ϕ becomes

$$\phi_0 = 1 + \alpha\eta - e^{\alpha\eta} \quad (29)$$

As usual, only terms containing k and S are retained in the second approximation. The equation to be solved is

$$\tilde{\mu}\phi'''' + 2\frac{d\tilde{\mu}}{d\eta}\phi''' + \frac{d^2\tilde{\mu}}{d\eta^2}\phi'' = ikR[(\overline{U} - c_0)\phi_0'' - \overline{U}''\phi_0] \quad (30)$$

or

$$\phi_1'''' - 2\alpha\phi_1''' + \alpha^2\phi_1'' = ikR[(\overline{U} - c_0)\phi_0'' - \overline{U}''\phi_0]e^{\alpha\eta} \quad (31)$$

The boundary conditions are

$$(1)\phi_1'(0) = 0 \quad (32)$$

$$(2)\phi_1(0) = 0 \quad (33)$$

$$(3)\phi_1''(1) + \frac{\overline{U}''|_{\eta=1}}{c_0'}\phi_1(1) - \frac{\overline{U}''|_{\eta=1}c_1}{(c_0')^2}\phi_0(1) = \frac{R\Sigma}{e^{-\alpha}}\frac{\phi_0(1)}{c_0'} \quad (34)$$

$$(4) - \left\{ k\left[\frac{\left(\dfrac{R\cos\theta}{F^2} + k^2RS\right)}{c_0'} - R\overline{U}'(1) - \frac{R\Pi}{c_0'}\right]\right\}\phi_0(1) + kRc_0'\phi_0'(1) - ie^{-\alpha}\phi_1'''(1) + i\alpha e^{-\alpha}\phi''(1) = 0 \quad (35)$$

where terms of order $k\phi_1$, $\phi_1\Sigma$, and $c_1\Sigma$ are neglected. The general solution of Equation (31) may be expressed as the sum of a complementary solution and a particular solution, after substitution of the appropriate $\overline{U}$, c_0, and ϕ_0:

$$\phi_1 = \overline{A} + \overline{B}\eta + \overline{C}e^{\alpha\eta} + \overline{D}\eta e^{\alpha\eta} + ikR(Ae^{2\alpha\eta} + B\eta e^{2\alpha\eta} + C\eta^2e^{2\alpha\eta} + De^{3\alpha\eta}) \quad (36)$$

The constants A, B, C, and D are determined, respectively, as

$$A = \frac{c_0 + 3B_2 - 3B_1\alpha - 7.5B_1}{4\alpha^2} \quad (37)$$

$$B = \frac{4B_1 + B_1\alpha - B_2}{4\alpha} \quad (38)$$

$$C = -\frac{B_1}{4} \quad (39)$$

$$D = \frac{B_1}{18\,\alpha^2} \quad (40)$$

Since ϕ_0 contains the constant 1 which is set arbitrarily, and since the coefficient of $\overline{A}$ can be shown to be ϕ_0 only, $\overline{A}$ in ϕ_1 can be set equal to zero. By applying Equations (31) to (35), and after long and tedious substitution and manipulation, $\overline{B}$, $\overline{C}$, $\overline{D}$, and c_1 can be determined, respectively, as

$$\overline{B} = -\alpha\overline{C} - \overline{D} - ikR(2\alpha A + B + E\alpha D) \quad (41)$$

$$\overline{C} = -ikR(A + D) \quad (42)$$

$$\overline{D} = \frac{i}{\alpha^2}\left\{ kR\left[\frac{\dfrac{\cos\theta}{F^2} + k^2S}{c_0'} - \overline{U}'(1) - \frac{\Pi}{c_0'}\right]\right.$$

$$(1 + \alpha - e^{\alpha}) - kRc_0'\alpha(1 - e^{\alpha}) - kRe^{-\alpha}[8\alpha^3e^{2\alpha}(A + B + C) + 12\alpha^2e^{2\alpha}(B + 2C) + 12\alpha Ce^{2\alpha} + 27\alpha^3De^{3\alpha}] + kR\alpha e^{-\alpha}[4\alpha^2e^{2\alpha}(A + B + C) + 4\alpha e^{2\alpha}(B + 2C) + 2Ce^{2\alpha} + 9\alpha^2De^{3\alpha}] \Bigg\} \quad (43)$$

$$c_1 = \frac{-R\Sigma}{e^{-\alpha}E} + ikRb_2 - ikRb_3\left(\frac{\cos\theta}{F^2} + k^2s - \Pi\right) \quad (44)$$

where

$$b_2 = \frac{c_0'}{E(1 + \alpha - e^{\alpha})}\Bigg\{4\alpha^2e^{2\alpha}(A + B + C) + 4\alpha e^{2\alpha}(B + 2C) + 2Ce^{2\alpha} + 9\alpha^2De^{3\alpha} - E(A + D) - E(2\alpha A + B + 3\alpha D) + E(Ae^{2\alpha} + Be^{2\alpha} + Ce^{2\alpha} + De^{3\alpha}) - (A + D)(\alpha^2e^{\alpha} - E - E\alpha + Ee^{\alpha}) - \frac{(2\alpha e^{\alpha} + \alpha^2e^{\alpha} - E + Ee^{\alpha})}{\alpha^3}\Bigg[\frac{\beta\alpha e^{\alpha}(1 + \alpha - e^{\alpha})}{(e^{\alpha} - 1)z_1} + c_0'\alpha(1 - e^{\alpha}) + 4\alpha^3e^{\alpha}(A + B + C) + 8\alpha^2e^{\alpha}(B + 2C) + 10\alpha Ce^{\alpha} + 18\alpha^3De^{2\alpha}\Bigg]\Bigg\} \quad (45)$$

$$b_3 = \frac{E - Ee^{\alpha} - 2\alpha e^{\alpha} - \alpha^2e^{\alpha}}{\alpha^2E} \quad (46)$$

$$E = \frac{\overline{U}''|_{\eta=1}}{c_0'} = \frac{\alpha^2e^{\alpha}}{1 + \alpha - e^{\alpha}} \quad \text{for} \quad \alpha \neq 0 \quad (47)$$

Since $\phi = \phi_0 + \phi_1$, the complete solution for ϕ is

$$\phi = 1 + (\alpha + \overline{B})\eta + (\overline{C} - 1)e^{\alpha\eta} + \overline{D}\eta e^{\alpha\eta} + ikR[Ae^{2\alpha\eta} + B\eta e^{2\alpha\eta} + C\eta^2e^{2\alpha\eta} + De^{3\alpha\eta}] \quad (48)$$

with $\overline{B}$, $\overline{C}$, $\overline{D}$ given by Equations (41) to (43), and A, B, C, D given by Equations (37) to (40). The complex wave velocity $c = c_0 + c_1 = c_r + ic_i$ is expressed as

$$c_r = \frac{(2 - \beta) + \beta\dfrac{1 + \alpha - e^{\alpha}}{e^{\alpha} - 1}}{z_1} - \frac{R\Sigma_r}{e^{-\alpha}E} - b_3kR\Pi_i \quad (49)$$

TABLE 1. VALUES OF b_2 AND b_3 FOR DIFFERENT VALUES OF α AND β

α	β	b_2	b_3
−3.0	0.0	0.376810	0.181497
−2.0	0.0	0.517131	0.216166
−1.0	0.0	0.757566	0.264241
0.0	0.0	1.200000	0.333333
0.2	0.0	1.329386	0.350690
0.4	0.0	1.478100	0.369522
0.6	0.0	1.649548	0.389989
0.8	0.0	1.847804	0.412269
1.0	0.0	2.077748	0.436564
2.0	0.0	3.955008	0.597264
3.0	0.0	8.259626	0.858188
0.0	1.0	0.000000	0.333333
0.2	1.0	−0.103984	0.350690
0.4	1.0	−0.202550	0.369522
0.6	1.0	−0.295638	0.389989
0.8	1.0	−0.383222	0.412269
1.0	1.0	−0.465307	0.436564

$$c_i = \frac{-R\Sigma_i}{e^{-\alpha}E} + b_2kR - b_3kR\left[\frac{\cos\theta}{F^2} + k^2S - \Pi_r\right] \quad (50)$$

where the perturbation stresses are complex and can be written as

$$\Sigma = \Sigma_r + i\Sigma_i, \quad \Pi = \Pi_r + i\Pi_i \quad (51)$$

Instability occurs when

$$\frac{-\Sigma_i}{e^{-\alpha}Ekb_3} + \Pi_r + \frac{b_2}{b_3} > \frac{\cos\theta}{F^2} + k^2S \quad (52)$$

For small k, or $k^2S << \cos\theta/F^2$, the above criterion becomes

$$F^2\left[\frac{-\Sigma_i}{e^{-\alpha}Ekb_3} + \Pi_r + \frac{b_2}{b_3}\right] > \cos\theta \quad (53)$$

In the limit of $\alpha = 0$, the velocity profile of the primary flow reduces to

$$\overline{U} = \frac{6(\beta - 1)\eta^2 + 6(2 - \beta)\eta}{4 - \beta} \quad (54)$$

and it can be shown that the instability criterion is (Yih, 1977)

$$\frac{3\Sigma_i}{2k} + \Pi_r + \left[\frac{144(1 - \beta)(2 - \beta)}{5(4 - \beta)^2}\right] > \frac{\cos\theta}{F^2} + k^2S \quad (55)$$

For small k, or $k^2S << \cos\theta/F^2$, the above criterion reduces to

$$F^2\left[\frac{3\Sigma_i}{2k} + \Pi_r + \frac{144(1 - \beta)(2 - \beta)}{5(4 - \beta)^2}\right] > \cos\theta \quad (56)$$

RESULTS AND DISCUSSION

Interfacial Shear

In the presence of interfacial shear only, the instability criterion is given, in dimensionless form, by Equation (55) or (56). When Σ_i, Π_r, and $\beta = 0$, this readily reduces to the Benjamin-Yih analysis

$$\frac{18}{5} > \frac{\cos\theta}{F^2} + k^2S \quad (57)$$

and, for $k \to 0$

$$R > \frac{5}{6}\cot\theta \quad (58)$$

When $\beta \neq 0$, the perturbation stresses Π and Σ have to be evaluated. In order to estimate these surface stresses, a shear flow model due to Miles (1957, 1962) and Benjamin (1959) can be used which assumes a mean dimensional velocity profile in the gas phase where turbulent fluctuations are ignored. The main results of the estimate are summarized in the paper by Craik (1966). Craik studied the generation of surface waves on a horizontal thin liquid film of water by airflow. When the gas and liquid are different from air and water, a dimensional scaling of the physical properties can be applied. Simple calculations show that Σ_i and Π_r are positive and are functions of the wave number of the disturbance and the physical and hydrodynamical parameters of the gas flow. Equations (55) and (56) therefore indicate that the presence of interfacial shear always creates additional instability through the perturbation stresses. Even when $\beta = 1$, the liquid film may still be unstable, depending on the relative magnitudes of Σ_i, Π_r, θ, and σ. This is in contrast to the situation of plane Couette flow found

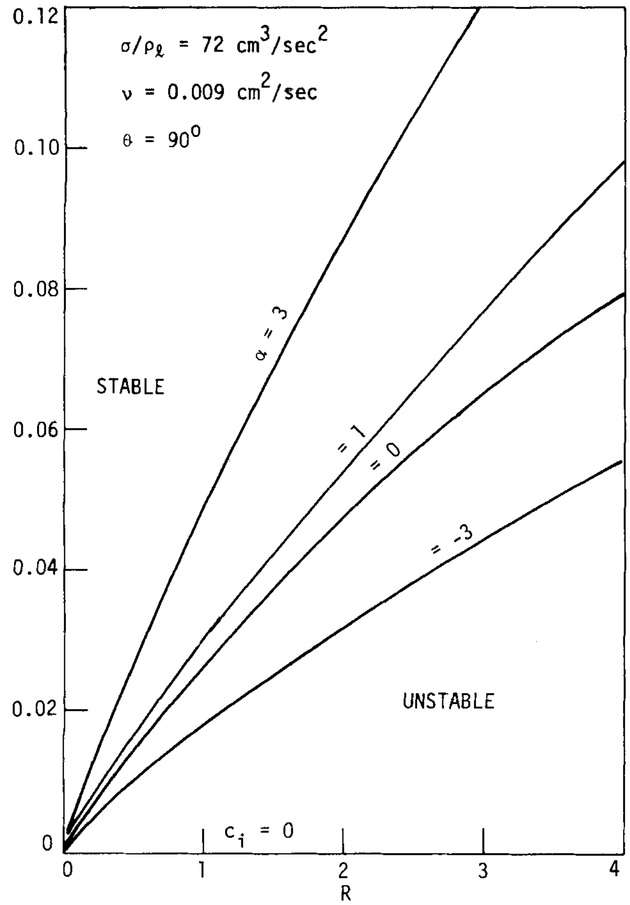

Fig. 1. Neutral stability curves for various α's in vertical film flow, $\theta = 90$ deg.

in confined geometries, where the flow is always stable. The analysis here can also be applied to the case of countercurrent gas flow by setting τ_1 negative. Then, β may become either positive or negative, depending on the sign of U_1. Equation (55) shows that in the presence of interfacial shear, the Froude number becomes the governing parameter which is different from the zero interfacial shear case, where the Reynolds number is

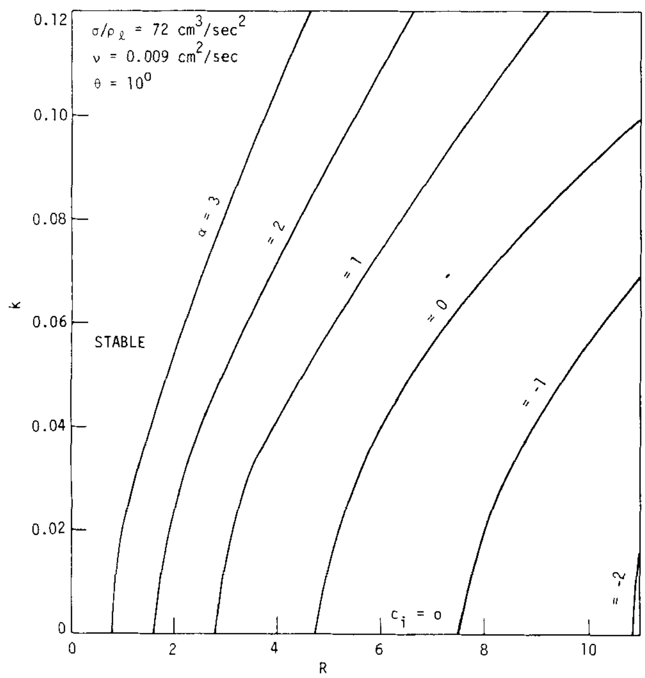

Fig. 2. Neutral stability curves for various α's, $\theta = 10$ deg.

the governing parameter. This is because when $\beta = 0$, a unique relation exists between the Reynolds number and the Froude number for a falling film but not for the cases $\beta \neq 0$. By converting Equation (55) into its dimensional form, the results of Smith (1970) and Craik (1966) are recovered.

Heat Transfer

The instability criterion is given by

$$\frac{b_2}{b_3} > \frac{\cos\theta}{F^2} + k^2 S \tag{59}$$

where b_2, b_3, and F are now functions of α only. Some typical values of b_2 and b_3 are shown in the first half of Table 1. It can be seen that an increase in α increases b_2, b_3, and the ratio b_2/b_3. Since the Froude number always increases for increasing α, Equation (59) indicates that an increase in α is destabilizing as compared to an isothermal liquid film. When α is positive, that is, $T_1 > T_0$ or wall cooling, the film is more unstable. When α is negative, that is, $T_1 < T_0$ or wall heating, the film becomes more stable, provided that natural convection effects are negligible. An explanation is that when $T_1 > T_0$, the liquid viscosity at the film surface will be less than the liquid viscosity at the wall or $\mu_1 < \mu_0$. If a reference temperature T_0 is used for an isothermal film, then a cooled film will be less viscous at the film surface than the corresponding isothermal film. Therefore, a cooled film will be more susceptible to surface wave formation and hence more unstable. The reverse is true for $T_1 < T_0$, in which $\mu_1 > \mu_0$; the film surface exhibits higher viscosity and so the surface is more resistant to external disturbances. Obviously, an increase in viscosity is a stabilizing factor in liquid film flow. In most experiments on film flow, it is much easier to vary the film Reynolds number than the Froude number. In order to compare the stability characteristics of liquid films accompanied by heat transfer and interfacial shear with the isothermal, zero, interfacial shear films, it is necessary to compare them under the same Reynolds number. For a given flow rate, the film thickness can be calculated from the momentum balance equation (Yih, 1977) if the physical properties of the liquid, the angle of inclination, α, and β (or τ_1) are known. Then, the Froude number can be evaluated for each different case of α and β. If Σ_i and Π_r are known or can be calculated from hydrodynamic parameters of the gas flow, the stability criterion Equation (52) can be compared with Equation (57) under the same film Reynolds number.

Neutral stability curves for a typical liquid (water) with a given kinematic surface tension and kinematic viscosity are shown in Figures 1 and 2 for two different angles of inclination: 90 and 10 deg. The axis $k = 0$ is also a part of the neutral stability curve. The upper part above each curve represents a stable region where infinitesimal harmonic disturbances are damped. When $S = 0$, a vertical falling film is always unstable for positive values of α. For θ less than 90 deg., bifurcation points are usually formed depending on the values of α. As θ is decreasing, the neutral stability curves shift to the right, and the critical Reynolds number R_{cr} also shifts to the right. It seems that a negative α shifts the R_{cr} more in magnitude than a positive α, as can be seen from Figure 2. The values of R_{cr} are obtained from Equation (59) by solving F_{cr} using trial and error. Obviously, heating of the wall is more stable than cooling, and the angle of inclination is a stabilizing factor. Figure 2 shows that at $\theta = 10$ deg practically high flow rates can be achieved without any surface wave formation. The possibility of achieving high flow rates increases with increased heating but decreases with increased cooling.

Table 2. Comparison of the Effect of Heat Transfer on Surface Wave and Shear Wave Instability

	Falling film flow	Plane Poiseuille flow (Potter and Graber, 1972)	Laminar boundary-layer flow (Wazzan et al., 1968)
Wave formation	Surface wave and shear wave, appear at small R	Shear wave, appears at large R	Shear wave, appears at large R
Modified O-S equation	Same	Same	Same
Boundary conditions	Influenced by heat transfer	Not influenced	Not influenced
Mean velocity profile	$\frac{d}{dY}\left(\mu \frac{dU}{dY}\right) = -\rho_1 g \sin\theta$, deviates from semiparabolic	$\frac{d}{dY}\left(\mu \frac{dU}{dY}\right) = \frac{dP}{dX}$, skew-symmetric	Solution of boundary-layer equations of momentum, deviates from Blasius profile
Solution method	Successive perturbation	Numerical integration	Numerical integration
Cooling of wall	Destabilizing	A temperature difference between the two walls is always destabilizing	Destabilizing
Heating of wall	Stabilizing		Stabilizing, but an optimum is found in which further heating decreases the R_{cr}
Causes of instability	Increased cooling produces inflection points in mean velocity profile, viscosity, gradient terms destabilizing	Viscosity gradient terms destabilizing	Increased cooling produces inflection points in mean velocity profile, viscosity gradient terms destabilizing

It is of interest to compare the effects of heat transfer on surface wave instability in falling liquid films with that of shear wave instability in laminar boundary-layer flow and plane Poiseuille flow. The flow of a falling liquid film can be visualized as one half of a plane Poiseuille flow with a free surface. Gravity and surface tension forces are dominant in falling films, whereas in plane Poiseuille flow, the pressure gradient is usually the driving force. While only shear waves appear in confined flows, both surface waves and shear waves may exist on a liquid film at small Reynolds number. The surface waves commonly observed usually have long wavelengths and are more unstable than short waves which are damped by surface tension. In the presence of heat transfer, the temperature gradient induces a viscosity variation across the flowing liquid, and additional viscosity gradient terms therefore appear in the O-S equation. A comparison of

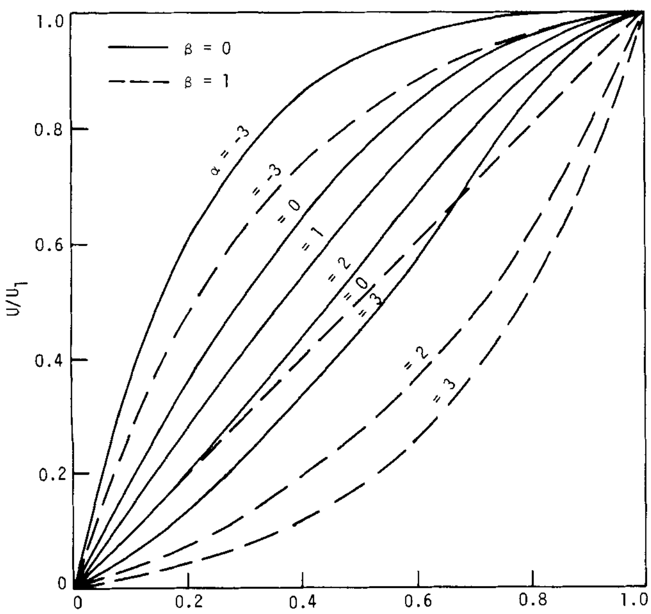

Fig. 3. Dimensionless velocity profiles as a function of α and β (Shair, 1971).

the effects of heat transfer on surface wave and shear wave instability is shown in Table 2. Although there is a fundamental difference between surface wave and shear wave instability, the effects of heat transfer on them are in many respects similar. A major cause of instability in these flows is the extra viscosity gradients which appear in the modified O-S equation. Another cause of instability is probably due to the modification of the mean velocity profile by heat transfer. As can be seen from Figure 3, at $\alpha = 2$ and $3 (\beta = 0)$, wall cooling produces an inflection point in the mean velocity profile, while wall heating does not produce this kind of inflection point. This type of phenomenon is also found in laminar boundary-layer flows. As pointed out by Potter and Graber (1972) and further illustrated in this study, although the viscosity gradient terms may be small, their effects cannot be neglected. It should also be noticed that the instability of falling film flow and laminar boundary-layer flow both depend on the direction of heat transfer, whereas the plane Poiseuille flow does not. This further indicates that the inflection points found in the mean velocity profile are a cause of instability. In the plane Poiseuille flow studied by Potter and Graber, water was used as the flowing liquid. A linear temperature difference was imposed, and they also used the exponential viscosity-temperature relationship as used in this study. The value of the activation energy for viscosity E_a was given as 1 741°K. Iyer's (1930) experimental correlation of eighty-seven liquids gave E_a as 1 832°K for water, and the E_a for other liquids ranged from a few hundred to 3 000°K. In a falling film of water, if T_0 is 300°K and the temperature drop $T_\alpha - T_0$ is 10°K, then this will give an α approximately equal to 0.2. If $T_1 - T_0$ is 100°K, then α is approximately 2. Therefore, the ranges of α reported in this study are sufficient for most practical operating conditions.

Both Heat Transfer and Interfacial Shear

Some of the values of b_2 and b_3 in Equation (52) are computed for $\beta = 1$ in the second half of Table 1. A contrast with the case of $\beta = 0$ is observed, and the values of b_2 and b_2/b_3 decrease with increasing α. The value of b_3 is independent of β and is only a function of α. It can be recalled that b_2/b_3 increases for increasing

α when $\beta = 0$ and decreases for increasing β when $\alpha = 0$. Thus the ratio b_2/b_3 may increase or decrease, depending on the relative magnitudes and signs of α and β. The first two terms of Equation (52) both increase for increasing α or β, but the third terms may decrease and even be negative at large values of β, which means that the destabilizing effects are not simply additive. All these imply that there is a certain interaction between heat transfer and interfacial shear. This probably arises from the mean velocity profile variation. Figure 3 shows that at $\beta = 0$, inflection points may be formed at high values of α, while at $\beta = 1$, the velocity profiles are symmetrical at, say, $\alpha = 3$ and -3. This kind of symmetry is also found for other α's, and there are no inflection points. This means that an increase in β leads to a more stabilizing velocity profile. The same is true for increased wall heating, which also leads to a more stabilizing velocity profile. From the above discussion, it can be concluded that concurrent and countercurrent interfacial shear are destabilizing factors because of the formation of surface perturbation stresses. Wall cooling is also a destabilizing factor because of the lower viscosity generated at the film surface. However, these two destabilizing effects are not simply additive and have some interactions between them, primarily through the mean velocity profile variation. Wall heating is a stabilizing factor provided that natural convection effects are negligible.

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NOTATION

a = displacement of liquid surface
A, B, C, D = integration constants in Equation (27) or (36)
A, B, C, D = constants defined by Equations (37) to (40)
b_2, b_3 = constants defined by Equations (45) and (46)
$B_1, B_2 = \dfrac{1-\beta}{(1+\alpha-e^{\alpha})z_1}, \dfrac{\beta}{(e^{\alpha}-1)z_1}$
c, c_r, c_i = dimensionless wave velocity, real, imaginary
c_0, c_1 = first approximation of wave velocity, second approximation
c_0' = $c_0 - \overline{U}(1)$
E_a = activation energy for viscosity
E = $\overline{U}''|_{\eta=1}/c_0 - \overline{U}(1)$
f = variable in stream function for pressure defined by Equation (6)
F = Froude number, $<U>/(g\Delta)^{1/2}$
g = gravitational acceleration constant
k = dimensionless wave number
$\hat{p}, p_1, p', \overline{P}$ = pressure, $\hat{p}/\rho_1<U>^2$, fluctuations, mean
R, R_{cr} = Reynolds number, $\Delta<U>\rho_1/\mu_0$, critical value
S = dimensionless surface tension, $\sigma/\rho_1<U>^2\Delta$
T, T_0, T_1 = temperature, at the wall (also the reference temperature for isothermal condition), at the gas-liquid interface
$\overline{T}, T'$ = mean temperature, fluctuations
$\hat{u}, u_1, u'$ = X component velocity, $\hat{u}/<U>$, fluctuations
$U, U_1, <U>, \overline{U}$ = X component velocity of primary flow, at gas-liquid interface, average, $U/<U>$
$\hat{v}, v_1, v'$ = Y component velocity, $\hat{v}/<U>$, fluctuations
x = X/Δ, dimensionless
X = axial distance of liquid film
Y = normal distance from the wall
z_1 = constant defined by Equation (3)

Greek Letters

α = $E_a(T_1 - T_0)/T_0^2$, dimensionless
β = interfacial shear parameter, $\tau_1\Delta/\mu_0 U_1\, e^{\alpha} - 1/\alpha$, dimensionless
Δ = average film thickness
η = dimensionless Y distance from the wall
θ = angle of inclination
$\mu, \mu_0, \mu_1, \tilde{\mu}$ = liquid viscosity, at T_0, at T_1, μ/μ_0, dimensionless
Π = $\tau_n/\rho_1<U>^2$, dimensionless
ρ_1 = liquid density
σ = surface tension
Σ = $\tau_t/\rho_1<U>^2$, dimensionless
τ = dimensionless time
τ_1 = constant interfacial stress
τ_n, τ_t = normal perturbation stress tangential
ϕ = variable in the stream function defined by Equation (5)
ψ = stream function for velocity

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